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A Mapping Technique for Efficient Random Event Generation with Constraints*

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A new technique for handling Monte Carlo integrations with δ-function constraints is presented. Points are generated in the unconstrained volume and then mapped by a scale transformation or other mapping onto the constrained subspace. The correct weight factors associated with such mappings are given for n-particle phase space with energy–momentum conservation constraints. The flexibility and simplicity of this scheme are discussed in comparison with existing alternative methods.

I. INTRODUCTION

Both the simulation of experiments and model theoretical analysis of multiparticle production processes in particle physics often involve phase space integrals of high dimensionality. Since it is not feasible to evaluate most such integrals analytically, Monte Carlo methods are frequently used and various techniques have been developed for specific purposes [1–4]. For integrals of high dimensionality, the integration volume usually increases rapidly with dimension and the number of random events needed for a desirable statistical accuracy may become prohibitive, unless the integrand is either sufficiently smooth (so that a very detailed sampling over the entire region of integration is not needed) or contributes appreciably to the integral only in a small portion of the region, where the random sample of events can be concentrated by a suitable choice of distribution. In essence, an efficient Monte Carlo method generates random events satisfying

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given constraints and concentrated in the regions where the integrand is important [5]. On the other hand, to achieve a high efficiency may become a formidable task in itself which can take a large amount of programming and computing time, possibly even exceeding the time needed for the actual calculation. Such an optimization scheme may also vary from case to case. Therefore, we almost always have to compromise between the efficiency, the total time, and the flexibility of the program, and it is desirable to have a sufficiently simple but versatile scheme, which can handle a variety of cases with reasonable efficiency and, of course, can be further optimized if needed.

Most existing event generators parametrize the n-particle phase space in such a way that the energy–momentum surface is mapped into some $(3n - 4)$-dimensional hypercube. The mapping is chosen so that a uniform density in the hypercube corresponds as nearly as possible to the desired density on the energy momentum surface. In this paper we present an alternative scheme in which a full $3n$-dimensional parametrization is used and each point generated is then mapped into a point on the energy–momentum surface. This allows much greater flexibility in the parametrization, allowing a closer simulation of the desired distribution. The procedure is not restricted to the use of variables defined in any special Lorentz frame.

In Sections II and III, we briefly summarize the basic concepts of phase space integrals and Monte Carlo integration. In Section IV, we discuss the techniques for satisfying the constraints by mapping events generated without the constraints onto the constrained subregion and discuss linear constraints as a simple example in Section V. Explicit energy–momentum conservation constraints are discussed in Section VI in terms of different sets of kinematical variables. In Section VII we discuss practical phase space event generators and in Section VIII we discuss the assignment of an arbitrary function. The techniques described here have already been successfully applied to multiparticle calculations [6–8].

II. Phase Space Integrals

Almost any measurable physical quantity in particle physics, whether a cross section, average value of some momentum or internal quantum number, etc., can be viewed as an integral (or sums of integrals) over an n-particle phase space of the form

$$\langle Q \rangle = \int P(p_1 \cdots p_n)\, Q(p_1 \cdots p_n)\, C(p_1 \cdots p_n) \prod_{i=1}^{n} d\Omega(p_i), \tag{II-1}$$

where

$$d\Omega(p) = d^4p\, \delta(p^2 - m^2)\, \theta(p_0) \tag{II-2}$$

is the invariant phase space volume element associated with a free particle of four-momentum p_μ and mass m [9].

$P(p_1 \cdots p_n)$ is the probability density, appropriately normalized, of observing an event specified by the set of momenta $p_1 \cdots p_n$. This probability density, of course, depends on the production mechanism and, in an experimental measurement, may also depend on the efficiency of the measuring apparatus.

$Q(p_1 \cdots p_n)$ is the kinematical quantity whose average is being measured (equal to unity if it is a cross section per invariant phase space volume or a total cross section).

$C(p_1 \cdots p_n)$ is a function representing kinematical constraints defining the measurement, consisting just of δ functions for total energy–momentum conservation if it is a total cross section but, in general, including angular or other kinematical cuts specifying a particular histogram bin or a particular geometry of the measuring apparatus.

Thus, in simulating an experiment or doing a theoretical analysis, we must in general evaluate an integral in a $3n$-dimensional phase space, constrained to at most a $(3n - 4)$-dimensional subspace. Formally the range of the integral is unrestricted, though the constraints C keep the integration region finite.

Besides the components of the four-momenta $p_{1\mu} \cdots p_{n\mu}$, there are alternative kinematical variables which are useful as variables of integration. We shall use the following notation for the kinematic variables associated with the four-momentum p^μ.

$$\mathbf{p} \equiv (p_x, p_y, p_z) \tag{II-3a}$$

is the three-momentum in rectangular coordinates;

$$\mathbf{p} \equiv (p, \theta, \varphi) \tag{II-3b}$$

is the three-momentum in polar coordinates;

$$\mathbf{p}_\perp \equiv (p_x, p_y) \tag{II-4}$$

is the transverse momentum, usually defined as perpendicular to the direction of the incident system in a production process;

$$p_0 \equiv (\mathbf{p}^2 + m^2)^{1/2} = (p_z^2 + \mu_\perp^2)^{1/2} \tag{II-5}$$

is the energy, where

$$\mu_\perp^2 \equiv \mathbf{p}_\perp^2 + m^2 \tag{II-6}$$

is the square of the transverse mass.

We define

$$p_\pm \equiv p_0 \pm p_z ; \tag{II-7}$$

then

$$\xi = \text{arc cosh}(p_0/m) \tag{II-8}$$

is the boost angle and

$$y = \text{arc cosh}(p_0/\mu_\perp) = \tfrac{1}{2}\ln(p_+/p_-) \tag{II-9}$$

is the rapidity. No more than three of these quantities listed are independent; if we denote these three as $\mathbf{x} = (x_1, x_2, x_3)$, then

$$d\Omega(p) = \Phi(x_1, x_2, x_3)\, d^3x, \tag{II-10}$$

for example,

$$\Phi(p_x, p_y, p_z) = 1/2p_0, \tag{II-11}$$

$$\Phi(p, \theta, \varphi) = \mathbf{p}^2/2p_0, \tag{II-12}$$

$$\Phi(p_x, p_y, y) = \tfrac{1}{2}, \quad \text{etc.} \tag{II-13}$$

The integral (II-1) can then be rewritten as

$$F = \int f(\mathbf{x}_1, \ldots, \mathbf{x}_n)\, C(\mathbf{x}_1, \ldots, \mathbf{x}_n) \prod_{i=1}^{n} d^3x_i, \tag{II-14}$$

where $f(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is the product of P, Q and the Φ's and C specifies the constraints. More generally, the kinematical variables for each individual particle may be replaced by other variables; for example, the invariant energies of various subsystems of particles, the relative angles defined by the subsystem, etc., can also be used as the variables of integration, with an appropriate nonfactorizable Jacobian Φ.

III. Monte Carlo Integration

Consider an l-dimensional integral

$$F = \int_R f(\mathbf{r})\, d^l r; \qquad \mathbf{r} \equiv (r_1 \cdots r_l), \tag{III-1}$$

where f is some function and R some region in the l-dimensional space of $\mathbf{r}$. The Monte Carlo procedure is to select N points $\mathbf{r}^{(i)}$ in R at random with probability distribution $p(\mathbf{r})$

$$\int_R p(\mathbf{r})\, d^l r = 1. \tag{III-2}$$

Then (see [5]),

$$F = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(\mathbf{r}^{(i)}) \frac{1}{p(\mathbf{r}^{(i)})} . \tag{III-3}$$

The convergence of this expression to (III-1) can depend sensitively on the choice of $p(\mathbf{r})$. Ideally, $p(\mathbf{r})$ should be directly proportional to $f(\mathbf{r})$. In particular, it is obvious that if $f(\mathbf{r})$ contains a δ function, then unless $p(\mathbf{r})$ also contains the same δ function, F given by Eq. (III-3) will remain zero.

Complicated probability distributions $p(\mathbf{r})$ cannot easily be generated on a computer, so it is in general necessary to transform the variables of the original expression in Eq. (III-1) so that the desired p can be reasonably well approximated by a simple distribution in the new variables (such as uniform or Gaussian).

IV. Constraints

Integrals over phase space usually involve both δ function and θ function constraints. Let us first consider the former, and as a trivial example, to illustrate the principles involved, let us consider the integral

$$I = \int \int_{-\infty}^{+\infty} f(x, y)\, \delta(x^2 + y^2 - a^2)\, dx\, dy, \tag{IV-1}$$

where we integrate a function of two variables over the boundary of a circle of radius a. Note that, because of the δ function the integration region may be formally taken as infinite.

The most straightforward way of treating the integral would be to eliminate the y integration and write

$$I = \int_{-a}^{+a} f(x, \pm(a^2 - x^2)^{1/2}) \frac{dx}{2(a^2 - x^2)^{1/2}} . \tag{IV-2}$$

From the point of view of Monte Carlo integration this suffers the disadvantage that a uniform distribution of points in x yields a very nonuniform distribution of points on the circle, and also that both branches of the square root have to be considered explicitly.

A much better procedure is to use polar coordinates to write

$$I = \int \int_{-\infty}^{+\infty} f(r\cos\theta, r\sin\theta)\, \delta(r^2 - a^2)\, r\, dr\, d\theta$$
$$\int_0^{2\pi} f(a\cos\theta, a\sin\theta) \frac{d\theta}{2} . \tag{IV-3}$$

However, this depends on recognizing a "good" choice of variables for the region defined by the δ function, which frequently is not easy, particularly in a multi-dimensional case.

For Monte Carlo integration, there is an equivalent procedure which preserves this advantage but can more easily be generalized. We introduce a scaling variable λ. Then replacing x by λx and y by λy, we can write

$$I = \iint_{-\infty}^{+\infty} f(\lambda x, \lambda y)\, \delta(\lambda^2 x^2 + \lambda^2 y^2 - a^2)\, \lambda^2\, dx\, dy \tag{IV-4}$$

for any λ, since the integration range is independent of scale. Let $\sigma(z)$ be any normalized function:

$$\int \sigma(z)\, dz = 1. \tag{IV-5}$$

Then

$$I = \int_{-\infty}^{+\infty} I\sigma(\lambda)\, d\lambda \tag{IV-6}$$

$$= \int\int\int_{-\infty}^{\infty} f(\lambda x, \lambda y)\, \delta(\lambda^2 x^2 + \lambda^2 y^2 - a^2)\, \lambda^2 \sigma(\lambda)\, dx\, dy\, d\lambda. \tag{IV-7}$$

Now carry out the λ integration using the δ function:

$$I = \int\int_{-\infty}^{\infty} f(\lambda_0 x, \lambda_0 y)\, \frac{\lambda_0 \sigma(\lambda_0)}{2(x^2 + y^2)^{1/2}}\, dx\, dy, \tag{IV-8}$$

where $\lambda_0 = a/(x^2 + y^2)^{1/2}$ is the value of λ causing the argument of the δ function to vanish. Thus we have retained the form of a two-dimensional integral but each point has been mapped onto the boundary of the circle. A uniform density in the (xy) plane results in a uniform distribution on the circle. As a Monte Carlo procedure the efficiency is determined by the form of the function σ, whether the set of values of λ_0 generated by the chosen distribution of values of x and y is a good set for evaluating (IV-8). The choice of σ will be discussed in Section VIII.

For a two-dimensional integral this is a somewhat heavy-handed procedure; however, it has the great advantage that the zero of the δ function argument has only to be determined numerically, rather than analytically. The resulting simplification may even, in this case, compensate for the fact that more points may be needed than by using the analog of (IV-3).

Let us now generalize this to the case of an l-dimensional integral[1].

Consider

$$I(a) = \int_{R_x} f(\mathbf{x})\, \delta[\varphi(\mathbf{x}) - a]\, d^l\mathbf{x}, \tag{IV-9}$$

[1] The procedure introduced here is closely related to what is known as "conditional Monte Carlo" [5].

where R_x denotes some region in the l-dimensional space, φ is a specified function, and a is a parameter.

Let R_y be another l-dimensional region (not necessarily different from R_x) and let

$$\mathbf{x} = \mathbf{g}\{\mathbf{y}; b\} \tag{IV-10}$$

be a family of transformations labeled by a parameter b which maps R_y into R_x for all values of b lying in some range. Let

$$J_\gamma(\mathbf{y}, b) = \| \partial \mathbf{g}\{\mathbf{y}; b\}/\partial \mathbf{y} \| \tag{IV-11}$$

be the transformation Jacobian. Then evidently

$$I(a) = \int_{R_y} f(\mathbf{g}\{\mathbf{y}; b\})\, \delta[\varphi(\mathbf{g}\{\mathbf{y}; b\}) - a]\, J_\gamma\{\mathbf{y}, b\}\, d^l\mathbf{y}. \tag{IV-12}$$

(We can in fact allow R_y to depend on b. $I(a)$ will still be independent of b if the region R_y, for all values of b in β, contains the entire set of values of y for which the argument of the δ function vanishes.) Let $\sigma(b)$ be any function of b with normalization

$$\int_\beta \sigma(b)\, db = 1. \tag{IV-13}$$

Then

$$\begin{aligned} I(a) &= \int_\beta I(a)\, \sigma(b)\, db \\ &= \int_\beta \int_{R_y} f(\mathbf{g}\{\mathbf{y}; b\})\, \delta[\varphi(\mathbf{g}\{\mathbf{y}; b\}) - a]\, J_\gamma(\mathbf{y}, b)\, d^l\mathbf{y}\sigma(b)\, db. \end{aligned} \tag{IV-14}$$

Eliminating the δ function by integrating over b and setting

$$J_\delta(\mathbf{y}, b) = J_\gamma(\mathbf{y}, b)[\partial\varphi(\mathbf{g}\{\mathbf{y}; b\})/\partial b]^{-1}, \tag{IV-15}$$

we obtain

$$I(a) = \int_{R_y} f(\mathbf{g}\{\mathbf{y}; b\})\, J_\delta(\mathbf{y}, b_0)\, \sigma(b_0)\, d^l\mathbf{y}, \tag{IV-16}$$

where $b_0 = b_0(\mathbf{y})$ is the solution of

$$\varphi(\mathbf{g}\{\mathbf{y}; b\}) = a, \tag{IV-17}$$

assumed to be unique and lie in β. The Monte Carlo result is then

$$I(a) = \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{i=1 \\ \mathbf{y}_i \in R_y}}^{N} \left\{ \frac{f(\mathbf{g}\{\mathbf{y}_i\,; b_i\})\, J_\delta(\mathbf{y}_i\,, b_{0i})\, \sigma(b_{0i})}{p(\mathbf{y}_i)} \right\}, \tag{IV-18}$$

where p is the probability density in R_y according to which the points y_i have been selected. Note that we have not only removed the δ function but also have the freedom to choose the function $\sigma(b)$ so as to improve convergence. Generalization to more than one δ-function constraint is straightforward. In Appendix A, we shall discuss a more general scheme and also an alternative to it, where more degrees of freedom in choosing an arbitrary function σ are allowed in both cases.

As discussed in the two-dimensional example, we have retained the form of an l-dimensional integral but each point has been mapped onto the subspace specified by the δ-function constraints. On the other hand, if the δ functions are directly eliminated, the resulting integral is less than l dimensional. However, an integral with a higher dimension may not need more points to calculate than one with a lower dimension, provided that a more efficient Monte Carlo scheme can be achieved for the former. It is usually difficult to find a "good" choice of variables for both solving the δ functions and sampling the subspace specified by the constraints. After using the transformation, the δ function is solved in terms of the transformation parameter and sampling is on the unconstrained variables $\mathbf{y}$. The resulting simplicity then may compensate for the need to perform a higher dimensional integral.

Typical transformation functions $\mathbf{g}$ are additive ($y_i \to y_i + b$), multiplicative ($y_i \to by_i$), or exponential ($y_i \to y_i^b$) which can be used to map the entire space $-\infty < y_i < +\infty$ into itself or, for some cases, $(0, \infty)$ or $(0, 1)$ into themselves.

Notice that in eliminating the δ function in Eq. (IV-14), we must solve Eq. (IV-17) for a given value of $a = a_0$, for any given $\mathbf{y}$. While it can be explicitly solved for some simple cases, it must be solved numerically for other cases. The most convenient way for the latter is by series expansion. In principle, Eq. (IV-17) can be inverted to give

$$b_0 = b(\mathbf{y}, a_0), \tag{IV-19}$$

though we may not be able to find an explicit form for $b(\mathbf{y}, a_0)$. Let $a_1 = \varphi(\mathbf{g}(\mathbf{y}, b_1))$ for some arbitrary value b_1. If b_1 is not too far from b_0, we can expand

$$b_0 = b_1 + \sum_{n=1}^{\infty} \frac{(a_0 - a_1)^n}{n!} \frac{\partial^n b}{\partial a^n}\bigg|_{a=a_1}. \tag{IV-20}$$

Although the derivatives $\partial^n b/\partial a^n$ can not be obtained by explicit differentiation, they are given in terms of $\partial^n a/\partial b^n$. Writing $b^{(n)} = (\partial^n b/\partial a^n)_{a=a_1}$ and $a^{(n)} = (\partial^n a/\partial b^n)_{b=b_1}$, we have

$$\begin{aligned} b^{(1)} &= [a^{(1)}]^{-1}; \\ b^{(2)} &= -[a^{(1)}]^{-3}\, a^{(2)}, \\ b^{(3)} &= [a^{(1)}]^{-5}\{3[a^{(2)}]^2 - a^{(1)}a^{(3)}\}, \\ b^{(4)} &= [a^{(1)}]^{-7}\{10a^{(1)}a^{(2)}a^{(3)} - 15[a^{(2)}]^{-3} - [a^{(1)}]^2\, a^{(4)}\}, \\ &\quad \ldots, \text{ etc.} \end{aligned} \tag{IV-21}$$

Rapid convergence of Eq. (IV-21) requires a good choice of b_1. In general, four terms should be plenty; if higher precision is needed, the output value of b_0 can be used as an input for another iteration. The choice of b_1, of course, varies from case to case and some examples will be discussed in Section VI.

Constraints involving θ functions can be treated by the usual methods; either the set of points is selected to directly satisfy the constraints or they can be selected from a larger region and then those points outside the domain of the θ function are discarded. Obviously, the latter procedure would give a null result for the δ-function constraints but only reduces the efficiency for the θ-function constraints. However, if a larger region is needed for each dimension, the efficiency decreases rapidly as the dimensionality increases. A trivial example is that if the range of integration is constrained to be inside a unit sphere while the points are uniformly selected inside a unit cube, then the ratio of the points inside the sphere to the total number of points rapidly decreases to zero as the dimensionality increases.

An alternative procedure, very similar to the one have discussed for the δ-function constraints, can also be worked out for the θ-function constraints. Again we use the mapping from R_y to R_x. For each point $\mathbf{y}$, solve for a range of b such that the mapped point satisfies the θ-function constraint. Then a value of b within this range can be randomly selected according to some probability distribution $\sigma(b)$ satisfying the normalization condition (IV-13). Mathematically, we have

$$\begin{aligned} I(a) &= \int_{R_x} f(\mathbf{x})\, \theta(\varphi(\mathbf{x}) - a)\, d^l\mathbf{x} \\ &= \int_{R_y} f(\mathbf{g}\{\mathbf{y}; b\})\, \theta(\varphi(\mathbf{g}\{\mathbf{y}; b\}) - a)\, J_\gamma(\mathbf{y}, b)\, d^l\mathbf{y} \\ &= \int_\beta \int_{R_y} f(\mathbf{g}\{\mathbf{y}; b\})\, \theta[\varphi(\mathbf{g}\{\mathbf{y}; b\} - a]\, J_\gamma(\mathbf{y}; b)\, \sigma(b)\, db\, d^l y, \end{aligned} \tag{IV-22}$$

and regard the θ function in the last line of Eq. (IV-22) as a function of b for any point $\mathbf{y}$. Once again, we may have to solve for the range of b numerically. Depending on the accuracy of the numerical solution, points may be accidentally selected outside the domain of the original θ function. Thus, if desired, we can explicitly test the constraint for each selected point and throw away these accidental ones.

V. Linear Constraints

A simple application of the method described in the last section is to the case of linear constraints. Let

$$F = \iint_{-\infty}^{\infty} f(\mathbf{x})\, \delta(\boldsymbol{\alpha} \cdot \mathbf{x} - a)\, d^l\mathbf{x}\ ; \tag{V-1}$$

then in the notation of Section IV,

$$\phi(\mathbf{x}) = \boldsymbol{\alpha} \cdot \mathbf{x} \tag{V-2}$$

$$R_x \equiv R_y = [-\infty, \infty]^l \tag{V-3}$$

The transformations we consider are linear

$$\begin{aligned} g\{\mathbf{y}; b\} &= \mathbf{y} + b\mathbf{u}, \\ \mathbf{u} &= (1, 1, 1 \cdots 1), \end{aligned} \tag{V-4}$$

from which it immediately follows that

$$J_\gamma(\mathbf{y}, b) = 1, \tag{V-5}$$

$$J_\delta(\mathbf{y}, b) = 1/(\boldsymbol{\alpha} \cdot \mathbf{u}), \tag{V-6}$$

$$\begin{aligned} b_0(\mathbf{y}) &= (\boldsymbol{\alpha} \cdot \mathbf{u})^{-1} [a - \boldsymbol{\alpha} \cdot \mathbf{y}], \\ &= a' - \boldsymbol{\alpha}' \cdot \mathbf{u}. \end{aligned} \tag{V-7}$$

Hence

$$F = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{f(\mathbf{y}^{(i)} + (a' - \boldsymbol{\alpha}' \cdot \mathbf{y}^{(i)})\, \mathbf{u})\, \sigma(a' - \boldsymbol{\alpha}' \cdot \mathbf{y}^{(i)})}{(\boldsymbol{\alpha} \cdot \mathbf{u})\, \rho(\mathbf{y}^{(i)})}, \tag{V-8}$$

where $\rho(\mathbf{y}^{(i)})$ is the probability density with which $\mathbf{y}^{(i)}$ are selected from R_y.

If the components y_j are generated independently with means μ_j and variances σ_j^2, i.e.,

$$\langle y_j \rangle = \mu_j, \tag{V-9}$$

$$\langle (y_j - \mu_j)(y_k - \mu_k) \rangle = \sigma_j^2 \delta_{jk}, \tag{V-10}$$

and if the μ_j satisfy the constraint

$$\Sigma\, \alpha_j \mu_j = a, \tag{V-11}$$

then the points $\mathbf{x}$ at which the function f is sampled have mean and second moments

$$\langle x_j \rangle = \langle y_j \rangle = \mu_j, \tag{V-12}$$

$$\langle (x_j - \mu_j)(x_k - \mu_k) \rangle = \sigma_j^2 \delta_{jk} - \alpha_j' \sigma_j^2 - \alpha_k' \sigma_k^2 + \Sigma \alpha_s'^2 \sigma_s^2, \tag{V-13}$$

where $a' = a/(\boldsymbol{\alpha} \cdot \mathbf{u})$ $\boldsymbol{\alpha}' = \boldsymbol{\alpha}/(\boldsymbol{\alpha} \cdot \mathbf{u})$ as before. In particular, if $\alpha = (1, 1, 1 \cdots 1)$, as occurs in the generation of linear momentum components, we have

$$\langle (x_j - \mu_j)(x_k - \mu_k) \rangle = \sigma_j^2 \delta_{jk} - (1/l)[\sigma_j^2 + \sigma_k^2 + \langle \sigma^2 \rangle], \tag{V-14}$$

where $\langle\sigma^2\rangle = (1/l)\sum_{s=1}^{l}\sigma_s^2$ is the average of all the σ_j^2. Hence the correlation introduced is of order $1/l$, and is therefore as small as consistent with the constraint. This method requires much less computation than that of Kittel *et al.* [12], produces the same result for identical Gaussian distributions and is much more general.

VI. Energy and Momentum Conservation Constraints

The methods described in the last two sections can now be applied to the general phase space integral (II-14). Suppose that the only constraints with which we are concerned are those of overall energy and momentum conservation:

$$\sum_{i=1}^{n} p_\mu(\mathbf{x}_i) = P_\mu \tag{VI-1}$$

so that

$$C(\mathbf{x}_1 \cdots \mathbf{x}_n) = \delta^{(4)}\left[\sum_{i=1}^{n} p_\mu(\mathbf{x}_i) - P_\mu\right].$$

We first generate a set of $3n$ variables $\{x_i\}$ by some random process. They will not satisfy (VI-1) in general. We apply four successive transformations to $\{x_i\}$, each depending on one parameter λ_a to give a new set

$$x_i' = x_i'[\{x_j\}, \lambda_1, \lambda_2, \lambda_3, \lambda_4] \tag{VI-2}$$

where $\lambda_1 \cdots \lambda_4$ are determined for each set of $\{\mathbf{x}_j\}$ so that the transformed $\{x_i'\}$ satisfy (VI-1). There are many ways of doing this; in this section we give the explicit formulation for three typical cases.

In each of these cases the equations for three of the parameters λ can be solved explicitly while the fourth must be solved numerically, according to the procedure described in Section IV.

(A) The simplest example assumes that we work in the center of mass system; $\mathbf{P} = 0$, $P^2 = s$. In this case we choose $\mathbf{p}_i$ as variables; the constraints being

$$\sum_{i=1}^{n} \mathbf{p}'_i = 0, \tag{VI-3a}$$

$$E(p') = \sum_{i=1}^{n} p'_{0i} = \sum_{i=1}^{n} (p_i'^2 + m_i^2)^{1/2} = (s)^{1/2} \tag{VI-3b}$$

associated with the transformations

$$\mathbf{p}_i' = \eta^{1/2}(\mathbf{p}_i + \boldsymbol{\lambda}), \tag{VI-4}$$

where $\boldsymbol{\lambda} = -(1/n)\sum \mathbf{p}_i$ and η is determined from (VI-3b). To do this we need the derivatives of E:

$$\frac{\partial^n E}{\partial \eta^n} = \frac{(2n-1)!!}{(-2)^{n-1}} \sum_{i=1}^{n} \frac{(p_i'^2/n)^n}{p_{0i}'^n}. \tag{VI-5}$$

A convenient starting point is

$$\eta_1 = [s - (\Sigma m_i)^2]/[E^2(\mathbf{p} + \boldsymbol{\lambda}) - (\Sigma m_i)^2]. \tag{VI-6}$$

(B) If we deal with a model with limited transverse momentum then we replace (VI-4) by

$$\left.\begin{aligned} \mathbf{p}_\perp{}' &= \mathbf{p}_\perp + \boldsymbol{\lambda}_\perp, \\ p_L{}' &= \eta^{1/2}(p_z + \lambda_z). \end{aligned}\right\} \tag{VI-7}$$

We then have the same solution as before with m_i^2 replaced by $\mu_{\perp i}^2 = m_i^2 + p_\perp^2$.

(C) An alternative scheme, which does not require $\mathbf{P} = 0$, but which also treats the longitudinal momenta separately is to use $\mathbf{p}_{\perp i}$ and y_i as variables. The constraints are then

$$\sum_{i=1}^{n} \mathbf{p}_{\perp i}' = \mathbf{P}_\perp, \tag{VI-8a}$$

$$\sum \mu_{\perp i}' \sinh y_i' = P_L, \tag{VI-8b}$$

$$\sum \mu_{\perp i}' \cosh y_i' = P_0, \tag{VI-8c}$$

where

$$\left.\begin{aligned} \mathbf{p}_{\perp i}' &= \mathbf{p}_{\perp i} + \boldsymbol{\lambda} \\ y_i' &= \alpha y_i + \beta \\ \mu_{\perp i}' &= (m_i^2 + p_{\perp i}'^2)^{1/2} \end{aligned}\right\} \tag{VI-9}$$

Clearly

$$\left.\begin{aligned} \boldsymbol{\lambda}_\perp &= -\frac{1}{n}\left(\mathbf{P}_\perp - \sum_{i=1}^{n} \mathbf{p}_{\perp i}\right), \\ \beta &= \ln\left(\sum_{i=1}^{n} \mu_{\perp i}' e^{\alpha y_i}/\mathbf{P}_+\right), \end{aligned}\right\} \tag{VI-10}$$

where α is the solution of

$$\left(\sum_{i=1}^{n} p_{i+}'\right)\left(\sum_{j=1}^{n} p_{j-}'\right) = s(\alpha) = P^2. \tag{VI-11}$$

We need the derivatives

$$\frac{\partial^l S}{\partial \alpha^l} = \sum_{r=0}^{l} \frac{l!}{r!\,(l-r)!} \left(\sum_{i=1}^{n} p_{i+} y_i{}^r \right) \left(\sum_{j=1}^{n} p_{j-} (-y_j)^{l-r} \right) \tag{VI-12}$$

and a starting value

$$\alpha = \frac{1}{y - y'} \ln \left(\frac{s}{\mu_\perp \mu_\perp{}'} \right), \tag{VI-13}$$

where $y' = \min(y_1 \cdots y_n)$, $y = \max(y_1 \cdots y_n)$ and $\mu_\perp{}'$, $\mu_\perp$ are the associated transverse masses.

Case (A) can be extended to deal with the case $\mathbf{P} \neq 0$. First use the procedure given above to generate a set $\{q_i\}$ with $\sum \mathbf{q}_i = 0$, $\sum q_{i0} = (P^2)^{1/2} = E$. Next construct

$$\mathbf{p}_{i\perp} = \mathbf{q}_i - ((\mathbf{q}_i \cdot \mathbf{P})\mathbf{P}/P^2). \tag{VI-14}$$

Then form

$$e_i = q_{i+} P_+ / (\mu_{i\perp} E). \tag{VI-15}$$

Set

$$\begin{aligned} p_{0i} &= (\mu_{i\perp}/2)(e_i + e_i^{-1}), \\ p_{zi} &= (\mu_{i\perp}/2)(e_i - e_i^{-1}); \end{aligned} \tag{VI-16}$$

the four vectors $\{(\mathbf{p}_{\perp i}, p_{zi}, p_{0i})\}$ have the required property.

VII. Procedure for Explicitly Evaluating Phase Space Integrals

Consider again the problem of calculating the expectation value of some measurable quantity in an N-particle state of total four momentum P_μ having a presented probability density per unit phase space volume. Rewriting Eq. (II-1) slightly, we wish to evaluate

$$\langle Q \rangle = \int Q(\mathbf{x})\, P(\mathbf{x})\, C(\mathbf{x}, P_\mu)\, d\Omega_N(\mathbf{x}), \tag{VII-1}$$

where $\mathbf{x}$ is a point in the $3N$-dimensional phase space Ω_N, Q is the measured quantity, P is the specified probability, and C are the constraint relations specifying the total four-momentum.

The calculational procedure is indicated schematically in the form of a flow chart in Fig. 1 and consists of the following stages.

A. Specify the parameters of the calculation. These include the number of particles N, and their rest masses, the total four-momentum P_μ, the numerical constants (e.g., means and variances) specifying the probability distribution according to which the $3N$ random variables determining a point in Ω_N are to be selected.

B. Pick a point $\mathbf{x}_0$ in the $3N$-dimensional space Ω_N according to some probability distribution function ρ and calculate the associated weight $w_0 = 1/\rho(\mathbf{x}_0)$.

C. Calculate the values of $b_i(\mathbf{x}_0, P_\mu)$, $i = 1,..., 4$, such that the point $\mathbf{x} = \mathbf{g}(\mathbf{x}_0, b_1 \cdots b_4)$ into which $\mathbf{p}_0$ is mapped by the transformation $\mathbf{g}$ lies in the $(3N - 4)$-dimensional subspace Ω_N corresponding to total four-momentum P_μ. Calculate the new point $\mathbf{x}$ and the corresponding Jacobian factor J_δ introduced in Section IV.

D. Calculate the probability densities $P(\mathbf{x})$ (associated with the point $\mathbf{x}$) and $\sigma(b_1 \cdots b_4)$ (associated with the parameters b_i); combine these to give an effective weight w for the event: $w = w_0 \cdot J_\delta P \cdot \sigma$.

E. Accumulate the weighted values $wQ(\mathbf{x})$ and the weight w together with other distributional information. Return to Step B if more points are needed.

F. Calculate $\langle Q \rangle = \Sigma wQ/\Sigma w$.

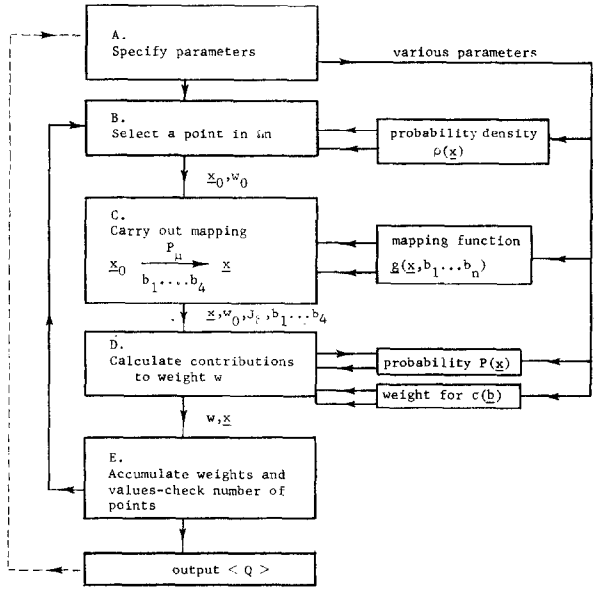

FIG. 1. Flow chart for event generator.

The dotted line in Fig. 1 indicates the important possibility of self-consistently determining some of the parameters. For example, if $k_1 \cdots k_{3N}$ are the independently generated random variables which determine the point $\mathbf{x}$, then the procedure can be used to determine the weighted means and variances of the k_i and these values can be then used as parameters in the distribution ρ so as to concentrate the events in regions of higher weight. In principle, the whole procedure could be incorporated within the framework of an adaptive Monte Carlo scheme allowing much more complex probability distributions.

More important, however, the dotted line indicates the step of adjusting the function $\sigma(b_1 \cdots b_4)$ so as to reduce the variance of the weights w. The method of doing this is discussed in detail in the next section; we note here that it requires that an adequate sample of weights be obtained before the parameters of the function σ can be reliably chosen.

So far we have discussed the type of program illustrated in Fig. 1, as if the program could be completely specified by defining the probability function P and measured quantity Q corresponding to the physical situation being considered, together with a finite number of parameters. In practice, of course, a number of very important choices have had to be made in the course of writing such a program, though to make an actual program more flexible a finite number of alternative options can be provided to be selected by setting an appropriate parameter.

The most important choices to be made, which can have a very large effect on the efficiency of the programs, are as follows.

(1) What $3N$ independent variables k_i should be generated as independent random variables to specify an event, and what probability distribution should be used to generate them ?

(2) What $3N$ variables x_i should be used to specify a point in the space Ω_N ?

(3) What functional form $\mathbf{g}(\mathbf{x}, b_1 \cdots b_4)$ should be used to specify the mapping onto the desired energy–momentum subspace of Ω_N ?

The choices under (1) are very broad. The simplest, of course, would simply be to take $k_i = x_i$, i.e., directly generate the phase space variables independently. But there are many other choices which may be useful where correlations are to be expected. For example, in multiperipheral models rapidity differences or momentum transfers may be more "natural" variables. If resonances are involved, certain effective mass combinations may be appropriate, etc. One respect, however, in which the procedure we describe is less flexible than some others is in the choice of probability distribution. As pointed out after (IV-13) the original distribution must be such that for *all allowed values* of the parameters b_1–b_4, the transformed region includes the whole of the subspace corresponding to P_μ. For this reason, rectangular or other compact distributions can sometimes give rise to problems, and must be treated with care.

We discussed in detail in Section VI some choices for the variables **x** and transformation functions **g**; there are, of course, many others. These choices will in turn suggest optimal choices for the arbitrary function $\sigma(b)$ discussed in Section VIII.

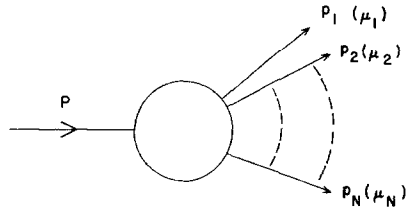

FIG. 2. Diagrammatic representation of standard event generation for N particles having total four momentum P_μ, individual four-momenta $p_{1\mu} \cdots P_{N\mu}$, and masses $\mu_1 \cdots \mu_N$.

The physical process corresponding to the event generator of Fig. 1 is indicated schematically in Fig. 2. So far we have implicitly suggested that the values of N, P_μ and the masses of the final state particles are fixed and then a large number of events is generated. However, there can also be circumstances in which any or all of these may vary from event to event. We list here some examples.

(a) *Inclusive Distributions.* If we want to compare with an experimental measurement, there is no reason why N itself cannot be chosen as a random variable provided that the other kinematic parameters are expressed as functions of N. The original probability distribution assumed for N can be modified in a self-consistent way as the calculation proceeds, so that the later events are distributed in accordance with the calculated multiplicity distribution.

(b) *Energy Dependence.* To obtain the dependence on the total energy, this can be varied from event to event. Alternatively the energy conservation constraint can simply be eliminated and the resulting events sorted into different energy bins. The energy spread of the resulting events will be determined by how wide a distribution is assumed for the original kinematic variables.

(c) *Resonance Formation.* Consider the reaction $A + B \rightarrow C_1^* + C_2 + \cdots C_n$, $C_1^* \rightarrow D_1 + D_2 \cdots + D_m$ illustrated in Fig. 3. The straightforward procedure is to regard this as an $(n + m - 1)$ particle state and choose the invariant mass of the particles D as a kinematic variable. An alternative procedure is to treat the first stage as an n-particle process where for each event the mass of C_1 is selected from the appropriate Lorentzian distribution, and then use the output four-momentum $p_{1\mu}$ as the input total four-momentum of the m-particle process corresponding to the decay. Further generalizations are possible. Note that in this case we have to keep track of two independent sets of weights and transformation parameters.

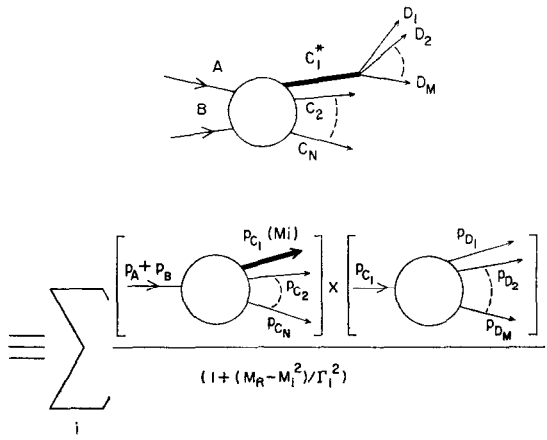

FIG. 3. Factoring of resonance production into two standard event generations, event generations, with resonance mass M_i randomly selected.

(d) *Cluster Formation.* Consider the reaction illustrated in Fig. 4: $A + B \rightarrow C_1 + C_2 \cdots C_n + D_1 + D_2 + \cdots D_m$, where we may wish to obtain a specific distribution in the four-momentum transfer q^2. In this case for each event the four-momentum q_μ can be selected from the desired distribution and then the procedure of Fig. 1 applied separately to the n-particle reaction involving the C's with total four-momentum $(p_{A\mu} + q_\mu)$, and the n-particle reaction involving the D's with total four-momentum $(p_B - q_\mu)$. Note that in this case as well as in case (c) there is an enormous advantage in having a scheme which works with arbitrary P_μ and is not tied to a specific Lorentz frame.

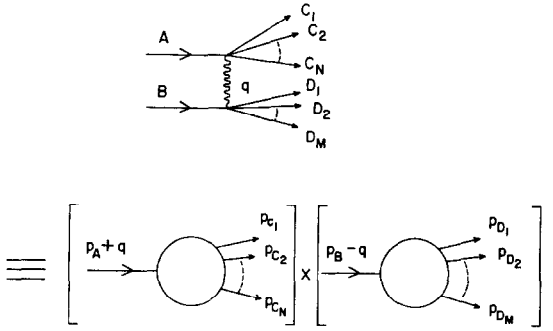

FIG. 4. Factoring of peripheral production at fixed four-momentum transfer into two standard event generations.

VIII. Assignment of Weights

In the previous section we outlined the procedure for calculating the expectation value of some function Q defined on phase space

$$\langle Q\rangle = \left(\sum_{i=1}^{N} w_i\sigma(\{b_i\})\, Q(\mathbf{x}_i)\right) \Big/ \sum_{j=1}^{N} w_j\sigma(\{b_j\}), \tag{VIII-1}$$

where i runs over a set of "events" generated as described above; w_i are the explicit weights (including the inverse probability of generating the unscaled point $\mathbf{x}_0$, the Jacobians for mapping to the point $\mathbf{x}$, and any model theoretical probability associated with the point $\mathbf{x}$); $\{b_i\}$ are the set of parameters specifying the mapping, and $\sigma(\{b_i\})$ is an arbitrary normalized function. We want to choose σ to optimize the estimate of $\langle Q\rangle$.

In general the choice of σ would depend on Q: we shall assume that we are interested in simulating an experiment, when the same set of events may have to be used to discuss many different averages. The best simulation (and in fact the best uniform procedure for estimating a general $\langle Q\rangle$) will come when the relative weights $w_i\sigma(\{b_i\})$ associated with different events are as close as possible, or in other words, we must choose σ so as to minimize

$$\Delta^2 = (\langle w^2\sigma^2\rangle - \langle w\sigma\rangle^2)/\langle w\sigma\rangle^2 \tag{VIII-2}$$

(In what follows we shall consider the case of a single b for simplicity: the argument goes through exactly when n b's determine an event if $\sigma(b)$ is replaced by $\sigma(\{b_\alpha\})$ and db by $\prod_1^n db_\alpha$.)

The averages in (VIII-2) are taken over the joint probability distribution of the pairs of values (w, b) occurring in event sample

$$\left.\begin{aligned}\langle w\sigma\rangle &= \int\int w\sigma(b)\,\rho(w, b)\,dw\,db,\\ \langle w^2\sigma^2\rangle &= \int\int w^2\sigma^2(b)\,\rho(w, b)\,dw\,db,\end{aligned}\right\} \tag{VIII-3}$$

$\rho(w, b)$ being the probability density. To solve for the function σ we take the functional derivative of Δ^2 with respect to σ and equate it to zero:

$$\begin{aligned}\frac{\delta\Delta^2[\sigma]}{\delta\sigma} &= \frac{\langle w\sigma\rangle^2 \cdot 2\sigma(b)\int w^2\rho(w, b)\,dw - 2\langle w\sigma\rangle\int w\rho(w, b)\,dw\,\langle w^2\sigma^2\rangle}{\langle w\sigma\rangle^4}\\ &= 0.\end{aligned} \tag{VIII-4}$$

Hence

$$\sigma(b) = (\bar{w}(b)/\overline{w^2}(b)) \cdot \langle w^2\sigma^2\rangle/\langle w\sigma\rangle, \tag{VIII-4}$$

where $\bar{w}(b) = \int w\rho(w, b)\, dw$ is the mean weight associated with a given b. The term involving the averages $\langle w^2\sigma^2\rangle$ and $\langle w\sigma\rangle$ is independent of b and can be ignored as an arbitrary normalization constant. (Although (VIII-4) is apparently an implicit equation for σ, taking the functional derivative at a different point $\sigma(b')$ and dividing will confirm that the normalizing term is irrelevant). Thus we obtain

$$\sigma(b) = \bar{w}(b)/\overline{w^2}(b) = \bar{w}(b)/(\delta^2(b) + \overline{w(b)^2}), \tag{VIII-5}$$

where $\delta^2(b)$ is the variance in w associated with a fixed value b. Since the variance in $w\sigma(b)$ comes both from the variance of means $\bar{w}(b)$ and the mean of the variances $\delta^2 b$, this ensures that if either component dominates the variance of w, it is eliminated in the variance of $w\sigma$, as we intend.

To evaluate (VIII-5) from observations on a finite sample of events we need to make some assumptions as to the form of $\rho(\omega, \mathbf{b})$. We first assume that $\ln w$ is normally distributed with mean $\mu(\mathbf{b})$ and variance $\gamma(\mathbf{b})$:

$$P(\ln w \mid \mathbf{b}) = (1/(2\pi\gamma(\mathbf{b}))^{1/2}) \exp -((\ln w - \mu(\mathbf{b}))^2/2\gamma(b)), \tag{VIII-6}$$

leading to the result

$$\sigma(\mathbf{b}) = \exp(-\mu(\mathbf{b}) + \tfrac{3}{2}\gamma(\mathbf{b})). \tag{VIII-7}$$

The assumption (VIII-6) is somewhat arbitrary; however it is plausible when we

[redacted]

tions. Reducing all of these ultimately to the generations of identically normally distributed random numbers, the central limit theorem suggests the assumption we have made.

If we now choose the definition of our scale parameters $b_1 \cdots b_4$ so that they are also normally distributed, then the standard results for multivariate normal distributions can be applied [10] and we can immediately assert that $\gamma(\mathbf{b})$ is independent of $\mathbf{b}$, and that therefore the second term in the exponent in (VIII-7) can be dropped as contributing an overall constant. To evaluate the other term we define

$$\left.\begin{aligned} \beta_i &= \langle b_i\rangle, \\ \Delta b_i &= b_i - \beta_i\,, \\ \mu &= \langle \ln w\rangle, \\ \Delta \ln w &= \ln w - \mu, \\ \sigma_{ij} &= \langle \Delta b_i\, \Delta b_j\rangle, \\ a_i &= \langle \Delta b_i\, \Delta \ln w\rangle, \end{aligned}\right\} \tag{VIII-8}$$

where the averages over the whole sample. Then

$$\mu(\mathbf{b}) = \mu + \sum_{i,j=1}^{s} \Delta b_i (\sigma^{-1})_{ij} a_j . \tag{VIII-9}$$

So that, again omitting overall constant terms, we use

$$\sigma(\mathbf{b}) = \exp\left\{ - \sum_{i,j=1}^{4} b_i a_j (\sigma^{-1})_{ij} \right\} . \tag{VIII-10}$$

For best results the distribution of b_i should be checked to see whether some other function of b_i has a distribution closer to normal.

Appendix A

In this appendix we discuss a more general scheme for eliminating the δ-function constraints by using the mapping (IV-10). The procedure discussed in Section IV is probably the simplest special case of this scheme; an alternative simple and useful case is given here.

Consider the one dimensional integral

$$I = \int_{r \in \rho} f(r)\, dr. \tag{A-1}$$

Suppose we can find a transformation

$$r = r(r', b), \qquad r' \in \rho', \quad b \in \beta, \tag{A-2}$$

which for fixed b maps ρ' into ρ, and for fixed r' maps β into ρ. (A scaling transformation over the range $[0, \infty]$ satisfies this condition, for example.) Let

$$J_{\rho'}(r', b) = \partial r / \partial r'; \qquad J_{\beta}(r', b) = \partial r / \partial b, \tag{A-3}$$

and let us introduce the notation $\tilde{f}(r', b) \equiv f(r(r', b))$ for any function f. Then we can rewrite I in two different ways:

$$I = I_{\rho'} = \int_{r' \in \rho'} \tilde{f}(r', b)\, J_{\rho'}(r', b)\, dr', \tag{A-4a}$$

$$I = I_{\beta} = \int_{b \in \beta} \tilde{f}(r', b)\, J_{\beta}(r', b)\, db, \tag{A-4b}$$

and hence, if $\sigma_{\rho'}(r')$, $\sigma_\beta(b)$ are normalized functions over the ranges ρ' and β, respectively, we can write

$$I = \int I_\rho \sigma_\beta(b)\, db = \int\int \tilde{f}(r', b)\, J_{\rho'}(r', b)\, \sigma_\beta(b)\, dr'\, db \tag{A-5a}$$

or

$$I = \int I_\beta \sigma_{\rho'}(r')\, dr' = \int\int \tilde{f}(r', b)\, J_\beta(r', b)\, \sigma_{\rho'}(r')\, dr'\, db. \tag{A-5b}$$

In this case r', b enter in a completely symmetrical way.

Now let us consider again the case of an l-dimensional integral, where now $\underline{x}$ is the integration variable over some l-dimensional R and we use the mapping (IV-10)

$$\mathbf{x} = \mathbf{g}\{\mathbf{y}; b\}. \tag{A-6}$$

Suppose we now pick a new set of coordinates

$$\begin{gathered} \mathbf{x} \equiv (r, \mathbf{w}); \qquad \mathbf{y} \equiv (r', \mathbf{w}) \\ r = r(r', b) \end{gathered} \tag{A-7}$$

where $\mathbf{w}$ is $(l-1)$ dimensional and is left invariant by the mapping (A-6). Then we have

$$I = \int_R F(\mathbf{x})\, d^l\mathbf{x} = \int F(r, \mathbf{w})\, J_\Omega(r, \mathbf{w})\, dr\, d^{l-1}\mathbf{w}, \tag{A-8}$$

where

$$J_\Omega(r, \mathbf{w}) = \| \partial(x_1 \cdots x_l)/\partial(r, w_1 \cdots w_{l-1}) \|$$

or in other words

$$\begin{gathered} I = \int I(\mathbf{w})\, d^{l-1}\mathbf{w}, \\ I(\underline{w}) = \int F(r, \mathbf{w})\, J_+(r, \mathbf{w})\, dr, \end{gathered} \tag{A-9}$$

and $I(\mathbf{w})$ is just a one-dimensional integral of the type considered above with $\mathbf{w}$ a parameter appearing at each stage of the integrand.

We now consider the case when $F(\mathbf{x})$ contains a δ-function constraint

$$F(\mathbf{x}) = f(\mathbf{x})\, \delta[\varphi(\mathbf{x}) - a]. \tag{A-10}$$

We carry out the substitutions leading to the equivalent of (A-5) and eliminate the δ functions by integrating over b, as in Section IV. We obtain

$$I = \int_{R_{\underline{y}}} f(\mathbf{g}\{\mathbf{y}, b_0\})\, J_\delta(\mathbf{y}, b_0)\, \sigma_\beta(b_0, \mathbf{w}(y))\, d^l\mathbf{y}, \tag{A-11a}$$

$$I = \int_{R_{\underline{y}}} f(g\{y, b_0\})\, J_\delta(\underline{y}, b_0) \frac{\partial r(r'(y)\, b)}{\partial b}\bigg|_{b=b_0} \frac{\partial r(\mathbf{r}'(\mathbf{y}), b_0)}{\partial r'}\bigg|_{r'(\underline{y})} \sigma_\rho(r'(\mathbf{y}), \mathbf{w}(\mathbf{y}))\, d^l\mathbf{y}. \tag{A-11b}$$

Equation (A-11a) is a slight generalization of (IV-16) allowing the arbitrary function σ_β to depend on the value of the invariant $\mathbf{w}$, and to be separately normalized for each $\mathbf{w}$; (A-11b) provides an alternative procedure.

As a result, Eq. (A-11a) involves an arbitrary function of b and $\mathbf{w}$ evaluated at $b = b_0$, while Eq. (A-11b) involves an arbitrary function of r' and $\mathbf{w}$, independent of the δ function. Thus the choice between these two procedures would depend on whether it is more convenient to find a suitable function for $\sigma(b, \mathbf{w})$ or $\sigma(r', \mathbf{w})$. In the latter case we would have to use a procedure analogous to that discussed in Section VIII to determine the optimal $\sigma(r', \mathbf{w})$.

Equation (A-11a) is more general than Eq. (IV-16) in the sense that σ can depend on the $l - 1$ invariants as well as b. However, we have to find a suitable set of invariants and a function σ which is normalized for all values of $\underline{w}$, while Eq. (IV-16) is much simpler and very often sufficient. Equation (A-11b) has an apparent advantage over Eqs. (IV-16) and (A-11a) that the choice of $\tilde{\sigma}$ directly depends on the distribution of $\underline{y}$ itself, which is chosen a priori, while for Eq. (A-11a), the choice for σ depends on the distribution of b_0, which has to be determined from the distribution of $\underline{y}$ and the δ function. However, in using Eq. (A-11b), we have to find a suitable set of variables r and r', as well as the invariants. A special case of Eq. (A-11b) is that $\tilde{\sigma}(r', \mathbf{w})$ is independent of the invariants in analogy with Eq. (IV-16). For the simple additive, multiplicative, or exponential transformation mentioned in Section IV, it is simple enough to find these variables and Eq. (A-11b) can be a very useful alternative. In particular for a multiplicative transformation the choice

$$r = \left(\sum_1^l x_1^2\right)^{1/2},$$

$$w_j = x_{j+1}/x_j, \qquad j = 1 \cdots l - 1,$$

is a suitable one, and is essentially that used by Kittel and Van Hove [2] in the procedure mentioned in the next appendix.

APPENDIX B: Comparison with Alternative Schemes for Monte Carlo Phase Space Integration

There exist many programs which generate events distributed in some manner within the Lorentz invariant n-particle phase space of $3n - 4$ dimensions. We comment briefly here on the different approaches which have been used and the differences from the method advocated here.

One major difference is that almost all rely on a set of random points generated uniformly inside a hypercube of $3n - 4$ dimensions. Their problem is then to find a mapping which maps the phase space volume (or, more precisely, which maps that part of phase space which contributes appreciably to the calculated quantity in question) into a significant fraction of that hypercube. By the mapping procedure introduced here we guarantee that every point lies in the phase space volume and for sufficiently high multiplicity we can fairly easily concentrate the majority of points in any desired subregion. The price we pay is that since in general the points prior to mapping must be generated in an infinite volume, they must be generated nonuniformly and a poor choice can lead to bad fluctuations, though these can be compensated to a large extent by the procedures of Section VIII.

The scheme described in this paper is not a specific event generator but a procedure for constructing a large class of such programs. It is therefore not possible to give any very specific results about time or efficiency since these will depend so much on the particular problem. One example, however, can be mentioned, as indicated in Section V this scheme can reproduce the same results as that of Kittel *et al.* [2] but with much less computation, the differential becoming more important with increasing multiplicity. Event generation rates of 1 msec/particle/event on a CDC 6600, have been attained using a multiregge matrix element, and obtaining events with small fluctuations of weights.

The most important advantage, however, is that the arbitrariness of the total four-momentum allowed by the scheme makes it possible to generate events which all satisfy some particular trigger (such as large transverse momentum), without having to project from a distribution over the entire phase space region. Most of the schemes discussed below cannot be used in this manner.

The other schemes fall principally into two classes. The first is especially appropriate at low energies when the relevant phase space volume is more or less isotropic (a good discussion is given in [1]). The variables used are obtained by representing the process as a sequence of two-body "decays." As the number n, of particles increases so does the number of alternative decay schemes, each of which can be represented by a tree graph. Two different schemes for the case $n = 5$ are illustrated in Fig. 3, and illustrate the fact that $(n - 1)$ vertices are needed. A two-body "decay" is characterized by the energy and angle associated with the "decay" products in the rest frame of the "decaying" system, usually expressed

as $(\mu_i, \theta_i, \varphi_i)$ where μ_i is the invariant mass of the decaying system, θ_i is the polar angle and φ_i the azimuth of one of the decay products relative to some fixed direction or, equivalently, (μ_i, t_i, ϕ_i) where t_i is the invariant four-momentum transfer of that decay product from one of the initial particles, whose direction serves as the axis about which ϕ_i is measured. The $(3n - 4)$ variables consist of the variables describing each of the $(n - 1)$ vertices, except the invariant mass μ_1 of the first vertex which is fixed by specifying the energy of the collision.

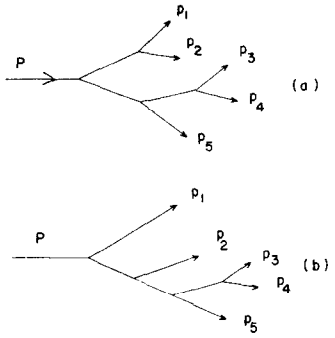

FIG. 5. Two alternative trees for $N = 5$.

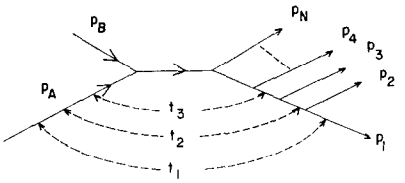

FIG. 6. Multiperipheral tree structure.

Since in principle an arbitrary distribution could be given to each of these variables and any choice of "tree" graph is allowed, the system allows great flexibility. However, since the range of values of μ_i depends on various μ_j $(j \leqslant i)$, the process of obtaining a desired distribution is rather complex unless the tree structure corresponds closely to the physical model of the reaction. Furthermore it is not very easy to simulate the behavior of high energy, multiparticle collisions, where transverse momenta are always small. To some extent this can be achieved (Friedman, Risk and Zang [1]; Byckling and Kajantie [3]) using the particular tree structure shown in Fig. 4 and using the four-momentum transfer variables t_i. By generating masses μ_i concentrated at the bottom of their allowed range, and damping the distribution in t, an efficient system is obtained for the multiperipheral model, but it is quite complex and not easily extendable to other models.

Kittel, Wojcik and Van Hove [2] have developed a scheme which is much more in the spirit of the one proposed here. They work with a phase space defined in terms of the $3n$ center of mass three-momenta $(p_L, \mathbf{p})$ separated into longitudinal and transverse components. In order to satisfy momentum conservation only, $(n-1)$ vectors are chosen independently, their components being Gaussianly distributed. Then these $(n-1)$ vectors together with the null vector are orthogonally transformed into n vectors whose sum is constrained to vanish and which are otherwise uncorrelated. This leaves only one constraint and setting

$$p_L \to \lambda p_L$$

which does not affect the vanishing of the sum $\sum p_L$, the momenta are transformed on to the appropriate energy–momentum surface. The present paper is essentially a generalization and extension of this scheme in two respects. First, the restriction to the particular variables in question has been removed so calculations can be carried out just as easily in rapidity or other variables and directly in any frame of reference desired. Second, the restriction to Gaussian distributions is removed for the transverse momenta since the cumbersome orthogonal transformation connecting independent and dependent variables is eliminated and the results of Section V used directly.

We finally mention the scheme developed by Pene and Kryzwicki [4]. To generate the transverse momenta they proceed in a way similar to Van Hove; then the longitudinal components are obtained from a multiperipheral type tree decomposition.

All of these schemes involve a lower dimension integration but more steps of computation and are less flexible than our simple, general scheme. On the other hand they may be more convenient in some special cases.

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